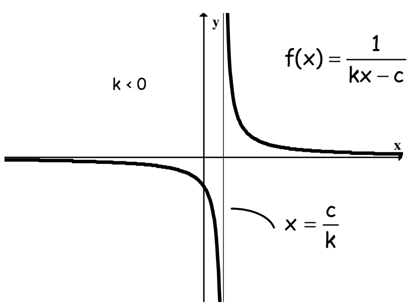
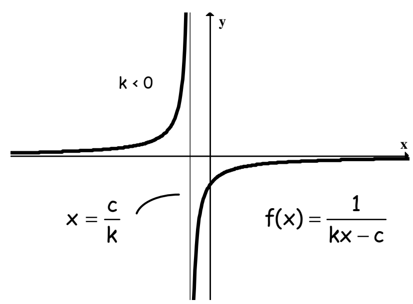
§3.1 RECIPROCAL OF A LINEAR FUNCTION

KEY CONCEPTS

- The reciprocal of a linear function has the form $\frac{1}{kx - c}$.
- The restriction on the domain of a reciprocal linear function can be determined by finding the value of x that makes the denominator equal to zero, that is, $x = \frac{c}{k}$
- The vertical asymptote of a reciprocal linear function has an equation of the form $x = \frac{c}{k}$.
- The horizontal asymptote of a reciprocal linear function has equation y = 0.
  - If k > 0, the left branch of a reciprocal linear function has a negative, decreasing slope, and the right branch has a negative, increasing slope.
  - If k < 0, the left branch of a reciprocal linear function has a positive, increasing slope, and the right branch has a positive, decreasing slope.





- $x \to a^+$ means "as x approaches a from the right"
- $x \to a^-$ means "as x approaches a from the left"
- The x-values that result in a zero in the numerator are the x-intercepts of the function, provided that the denominator is not zero.

Example:

1. Consider the function $f(x)=\dfrac{2}{4x-3}$.

a) State the domain and range.

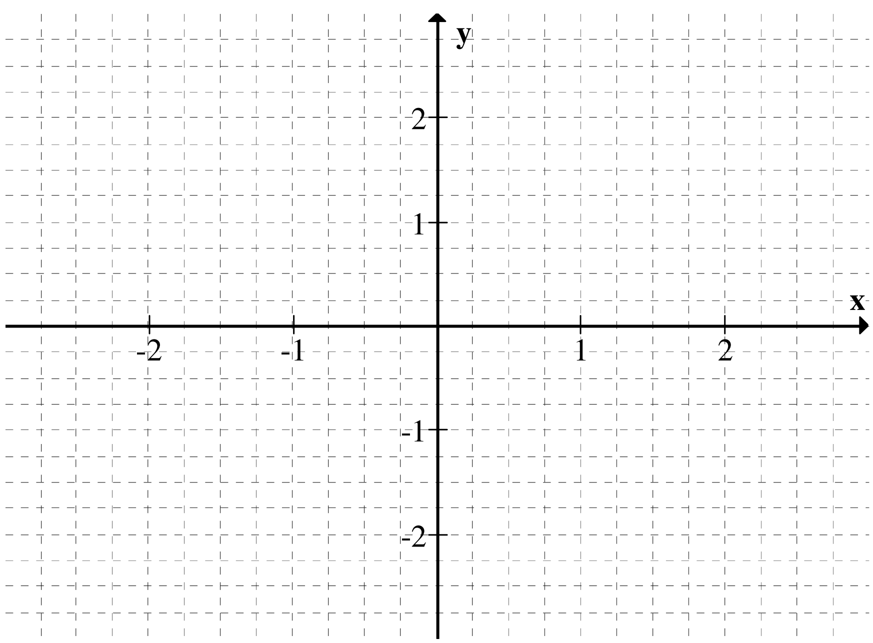
b) Write the equations of the asymptotes.

c) Determine the y-intercept.

c) Complete a table summarizing the behaviour of the function as it approaches the vertical asymptote and as it approaches $\pm\infty$.

As $x \rightarrow$	$f(x) \rightarrow$

d) Sketch a graph of the function and label all important parts.



§3.2 RECIPROCAL OF A QUADRATIC FUNCTION

KEY CONCEPTS

- Rational functions can be analysed using the following key features:
  - Asymptotes
  - Intercepts
  - Slope (positive or negative, increasing or decreasing)
  - Domain and Range
  - Positive and Negative Intervals
- Reciprocals of quadratic functions with two zeros have three parts, with the middle one reaching a maximum or minimum point. This point is equidistant from the two vertical asymptotes.
- The behaviour near asymptotes is similar to that of reciprocals of linear functions.
- All of the behaviours listed above can be predicted by analysing the roots of the quadratic relation in the denominator.

Examples:

1. Consider the function $f(x) = \dfrac{4}{4x^2 - 7x - 2}$.

   a) State the domain.

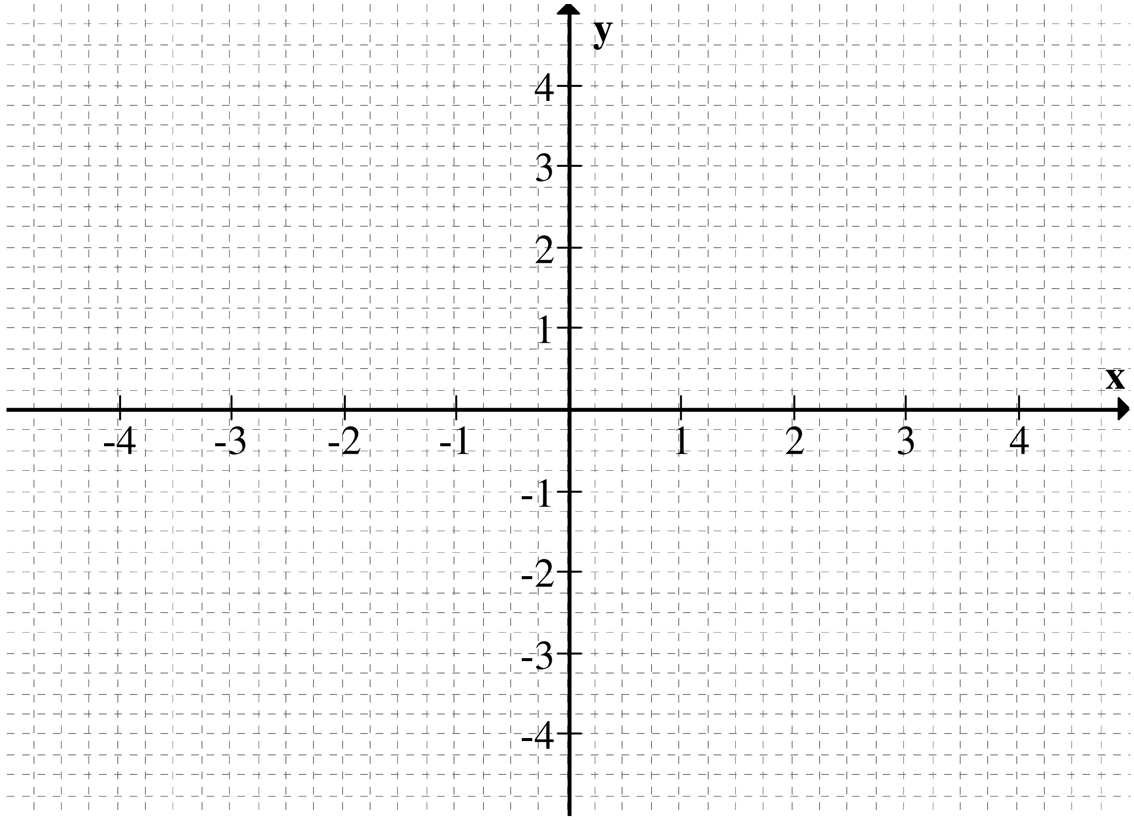
   b) Determine the equations of the asymptotes.

   c) Determine the y-intercepts.

   d) Complete a table to summarize the intervals of increase and decrease.

Interval					
Sign of f(x)					
Sign of Slope					
Change in Slope					

   d) Sketch a graph of the function and label all important parts.



e) State the range.

2. Each function described below is the reciprocal of a quadratic function. Write an equation to represent each function.

a) The horizontal asymptote is $y = 0$.
The vertical asymptotes are $x = -1$ and $x = 3$.
$f(x) < 0$ for the intervals $x < -1$ and $x > 3$.

b) The horizontal asymptote is $y = 0$.
There is no vertical asymptote (i.e. Domain = $\{ x \mid x \in \Re \}$)
The minimum point is (0, -0.25).

§3.3 RATIONAL FUNCTIONS OF THE FORM $f(x) = \frac{ax + b}{cx + d}$

KEY CONCEPTS

- A rational function of the form $f(x) = \frac{ax+b}{cx+d}$ has the following key features:
  - The equation of the vertical asymptote can be found by setting the denominator equal to zero and solving for x, provided the numerator does not have the same zero.
  - The equation of the horizontal asymptote can be found by dividing each term in both the numerator and the denominator by x and investigating the behaviour of the function as $x \to \pm\infty$.
  - The coefficient b acts to stretch the curve, but has no effect on the asymptotes, domain, or range.
  - The coefficient d shifts the vertical asymptote.
  - The two branches of the graph of the function are equidistant from the point of intersection of the vertical and horizontal asymptotes.

Examples:

1. Consider the function $f(x) = \frac{x-4}{3x-5}$.

   a) Determine the equation of the vertical asymptote.

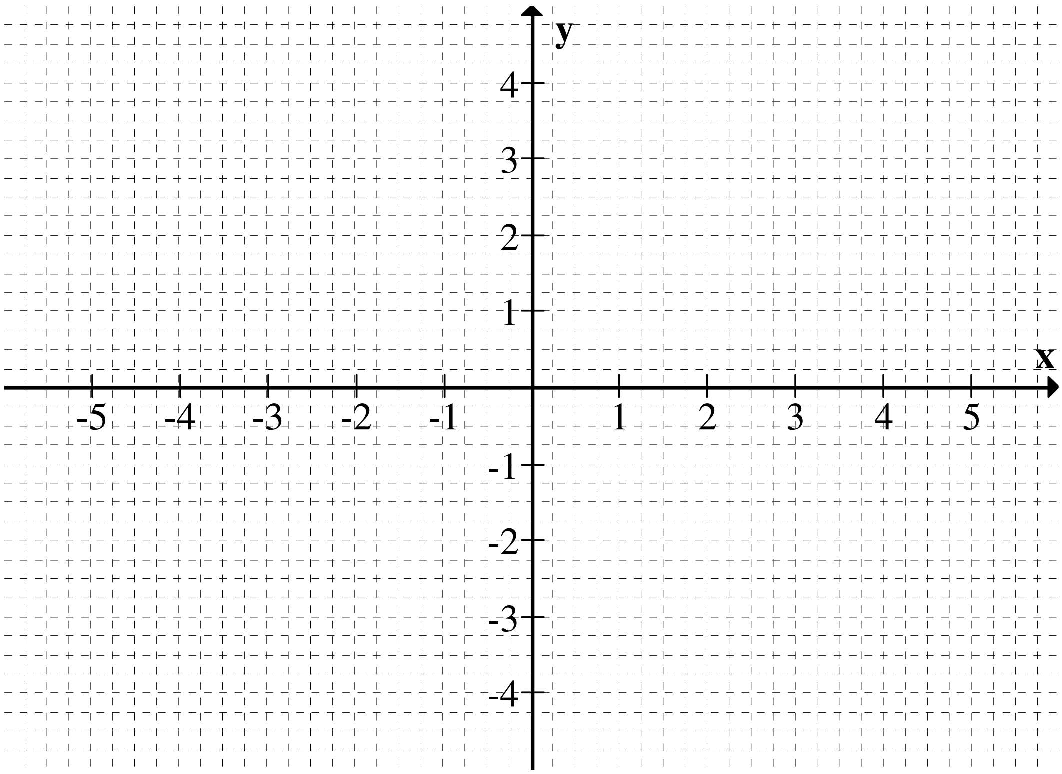
   b) Determine the equation of the horizontal asymptote.

   c) Determine the y-intercept.

d) Complete a table to summarize the intervals of increase and decrease.

Interval				
Sign of f(x)				
Sign of Slope				

d) Sketch a graph of the function and label all important parts.



e) State the domain and range.

2. Write an equation for a rational function whose graph has all of the indicated features.

- x-intercept of $\frac{4}{7}$
- horizontal asymptote at $y = \frac{7}{3}$
- vertical asymptote at $x = -\frac{2}{3}$
- y-intercept of -2

§3.4 SOLVE RATIONAL EQUATIONS AND INEQUALITIES

KEY CONCEPTS

- To solve rational equations algebraically, start by factoring the expressions in the numerator and denominator to find asymptotes and restrictions.
- Next, multiply both sides by the factored denominators, and simplify to obtain a polynomial equation. Then, solve using techniques you learned in Chapter 2.
- For rational inequalities:
  - It can often help to rewrite with the right side equal to 0. Then, use test points to determine the sign of the expression in each interval.
  - If there is a restriction on the variable, you may have to consider more than one case. For example, if $\frac{a}{x-k} < b$, case 1 is $x > k$ and case 2 is $x < k$.
- Rational equations and inequalities can be solved by studying the key features of the graph with paper or with the use of technology.
- Tables and number lines can help organize intervals and provide a visual clue to solutions.
- The **critical values of x** are those values where there is a vertical asymptote, or where the slope of the graph of the inequality changes sign.

Examples:

1. Solve and check by hand and/or using technology.

a) $\frac{3x}{3x+2} - \frac{3x}{3x-2} = 1$

b) $\frac{5x}{x-4}=\frac{3x}{2x+7}$

c) $\frac{x^2+9x+18}{x^2-5x+4}\rangle 0$

§3.5 MAKING CONNECTIONS WITH RATIONAL FUNCTIONS & EQUATIONS

KEY CONCEPTS

- When solving a problem, it's important to read carefully to determine whether a function is being analysed or an equation or inequality is to be solved.
- A full analysis will involve four components:
  1. Numeric (tables, ordered pairs, calculations)
  2. Algebraic (formulas, solving equations)
  3. Graphical
  4. Verbal (descriptions)
- When investigating special cases of functions, factor and reduce where possible. Indicate the restrictions on the variables in order to identify hidden discontinuities.
- When investigating new types of rational functions, consider what is different about the coefficients and the degree of the polynomials in the numerator and denominator. These differences could affect the stretch factor of the curve and the equations of the asymptotes and they could cause other discontinuities.
- **Discontinuities** are values at which a function becomes undefined. They may appear as asymptotes or as holes (or gaps).

Examples:

1. In order to create a saline solution, salt water with a concentration of 40 g/L is added at a rate of 500 L/min to a tank of water that initially contained 8000 L of pure water. The resulting concentration of the solution in the tank can be modelled by the function $C(t)=\dfrac{40t}{160+t}$, where C is the concentration, in grams per litre, and t is the time, in minutes.
   a) In how many minutes the saline concentration be 20 g/L?
   b) Is there an upper limit to the concentration in the tank? Explain.
   c) What restrictions must be placed on the domain of C if the tank has a maximum capacity of 120 000 L?

2. A company finds that its sales since the company started in 2000 can be modelled by the function $S(t)=\dfrac{20t^2+800t+300}{8t^2+10t+100}$, where S is the total sales, in millions of dollars, and t is the number of years since 2000.
   a) What were the sales in 2000?
   b) After many years, what does the model predict sales will be?
   c) Calculate the years when the sales are $9 million, algebraically.
   d) Use Technology Use technology to graph of the model. During what year were sales highest?
   e) If you were working in the human resources department for the company, would you recommend that the company hire more people based on this model? Explain your reasoning.

3. The weight (gravitational force) on a
100-kg object as a function of its height above mean sea level on Earth can be modelled by the formula $W(h)=\dfrac{4\times10^{16}}{\left(6.4\times10^{6}+h\right)^{2}}$, where *W* is the weight, in Newtons (1 kg weighs about 10 N) and *h* is the height above mean sea level, in metres.

a) How much does the object weigh at sea level?

b) If you were to take the object to the top of Mt. Everest (height 9000 m), what would its weight be?

c) How high would the object have to be to weigh 800 N? Round your answer to the nearest kilometre.

4. An integer *n* is squared, and the result doubled. Three is added to the same integer and the result squared. The ratio of the first answer to the second is then formed.

a) Write a function *R*(*n*) that gives the ratio of the two answers.

b) Sketch the graph of *R*.

c) A student claims that the value of *R* will always be less than 2. Is she correct? Explain.

d) Solve algebraically to determine the values of *n* for which $R(n) \le 0.5$. Illustrate your answer on a number line.

e) For which value(s) of *n* is $R(n) > 8$?